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On the coupled unsaturated–saturated flow process induced by vertical, horizontal, and slant wells in unconfined aquifers

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S1. Dimensionless transform for Eqs. (1)-(4)

For the purpose of mathematical convenience, we define the following dimensionless variables,

$$s_D = \frac{4\pi K d}{Q} s, t_D = \frac{K}{S_s d^2} t, x_D = \alpha_x \frac{x}{d}, y_D = \alpha_y \frac{y}{d}, z_D = \alpha_z \frac{z}{d}, x_{0D} = \alpha_x \frac{x_0}{d}, y_{0D} = \alpha_y \frac{y_0}{d}, z_{0D} = \alpha_z \frac{z_0}{d},\ \alpha_x = (K/K_x\ )^{1/2}, \alpha_y = \left(K/K_y\ \right)^{1/2},\ \alpha_z = (K/K_z\ )^{1/2}, K = \left(K_x K_y K_z\right)^{1/3}, u_D = \frac{4\pi K d}{Q} u, \kappa_D = \frac{d}{\alpha_z}\kappa,\ \beta = \frac{\kappa_D}{\sigma},\ \sigma = \frac{S_s d}{S_y \alpha_z},\quad b_D = \alpha_z \frac{b}{d}, \tag{S1}$$

where the subscript D denotes the dimensionless terms. Substituting above dimensionless variables into Eqs. (1)-(4), one obtains the following dimensionless forms of the governing equations for the saturated zone,

$$\frac{\partial^2 s_D}{\partial x_D^2} + \frac{\partial^2 s_D}{\partial y_D^2} + \frac{\partial^2 s_D}{\partial z_D^2} + 4\pi\delta(x_D - x_{0D})\delta(y_D - y_{0D})\delta(z_D - z_{0D}) = \frac{\partial s_D}{\partial t_D},\ 0 \le z_D < \alpha_z, \tag{S2a}$$

$$s_D(x_D, y_D, z_D, 0) = 0, \tag{S2b}$$

$$\frac{\partial s_D}{\partial z_D}(x_D, y_D, z_D, t_D)|_{z_D=0} = 0, \tag{S2c}$$

$$\lim_{x_D \to \pm\infty} s_D(x_D, y_D, z_D, t_D) = \lim_{y_D \to \pm\infty} s_D(x_D, y_D, z_D, t_D) = 0, \tag{S2d}$$

and for the unsaturated zone,

$$\frac{\partial^2 u_D}{\partial x_D^2} + \frac{\partial^2 u_D}{\partial y_D^2} + \frac{\partial^2 u_D}{\partial z_D^2} - \kappa_D \frac{\partial u_D}{\partial z_D} = \beta \frac{\partial u_D}{\partial t_D},\quad \alpha_z \le z_D < \alpha_z + b_D, \tag{S3a}$$

$$u_D(x_D, y_D, z_D, 0) = 0, \tag{S3b}$$

$$\frac{\partial u_D}{\partial z_D}(x_D, y_D, z_D, t_D)|_{z_D=\alpha_z+b_D} = 0, \tag{S3c}$$

$$\lim_{x \to \pm\infty} u_D(x_D, y_D, z_D, t_D) = \lim_{y \to \pm\infty} u_D(x_D, y_D, z_D, t_D) = 0, \tag{S3d}$$

and at the interface,

$$s_D - u_D = 0, \qquad z_D = \alpha_z, \tag{S4a}$$

$$\frac{\partial s_D}{\partial z_D} - \frac{\partial u_D}{\partial z_D} = 0, \qquad z_D = \alpha_z. \tag{S4b}$$

S2: Laplace domain solution of the unsaturated zone

The Laplace transformation of Eqs. (S3) and (S4) are written as

$$\frac{\partial^2 \bar{u}_D}{\partial x_D^2} + \frac{\partial^2 \bar{u}_D}{\partial y_D^2} + \frac{\partial^2 \bar{u}_D}{\partial z_D^2} - \kappa_D \frac{\partial \bar{u}_D}{\partial z_D} = \beta p \bar{u}_D, \qquad \alpha_z \leq z_D < \alpha_z + b_D, \tag{S5a}$$

$$\frac{\partial \bar{u}_D}{\partial z_D}(x_D, y_D, z_D, p)|_{z_D=\alpha_z+b_D} = 0, \tag{S5b}$$

$$\lim_{x_D \to \pm\infty} \bar{u}_D(x_D, y_D, z_D, p) = \lim_{y_D \to \pm\infty} \bar{u}_D(x_D, y_D, z_D, p) = 0, \tag{S5c}$$

and

$$\bar{s}_D - \bar{u}_D = 0, \qquad z_D = \alpha_z, \tag{S6a}$$

$$\frac{\partial \bar{s}_D}{\partial z_D} - \frac{\partial \bar{u}_D}{\partial z_D} = 0, \qquad z_D = \alpha_z, \tag{S6b}$$

where p is the Laplace transform parameter and the overbar indicates a variable in the Laplace domain. The Eq. (S5) in a cylindrical coordinate system can be written as following

$$\frac{1}{r_D}\frac{\partial \bar{u}_D}{\partial r} + \frac{\partial^2 \bar{u}_D}{\partial r^2} + \frac{\partial^2 \bar{u}_D}{\partial z_D^2} - \kappa_D \frac{\partial \bar{u}_D}{\partial z_D} = \beta p \bar{u}_D, \qquad \alpha_z \leq z_D < \alpha_z + b_D, \tag{S7a}$$

$$\frac{\partial \bar{u}_D}{\partial z_D}(r_D, z_D, p) = 0, \qquad z_D = \alpha_z + b_D, \tag{S7b}$$

$$\bar{u}_D(\infty, z_D, p) = 0. \tag{S7c}$$

Based on the methods of the separation variables and Eq. (5), the solution of Eq. (S7a) with boundary condition Eq. (S7c) will be (Rezaei et al., 2016):

$$\bar{u}_D(r_D, z_D, p) = \sum_{n=0}^{\infty} \frac{8\cos(\omega_n z_{0D})}{p\Psi(\omega_n)} K_0(\Omega_n |r_D - r_{0D}|) \mathcal{H}_n(z_D, p). \tag{S8}$$

Substituting Eq. (S8) into Eq. (S7) yields:

$$\frac{\partial^2 \mathcal{H}_n}{\partial z_D^2} - \kappa_D \frac{\partial \mathcal{H}_n}{\partial z_D} - (\beta p - \Omega_n^2)\mathcal{H}_n = 0, \qquad \alpha_z \leq z_D < \alpha_z + b_D, \tag{S9a}$$

$$\frac{\partial \mathcal{H}_n}{\partial z_D}(z_D, p) = 0, \qquad z_D = \alpha_z + b_D. \tag{S9b}$$

The general solution of Eq. (S9a) is

$$\mathcal{H}_n = \begin{cases} e^{Mz_D}[C_1 e^{Nz_D} + C_2 e^{-Nz_D}], & if \;\; \Delta > 0, \\ e^{Mz_D}[C_1 \sin(N_1 z_D) + C_2 \cos(N_1 z_D)], & if \;\; \Delta < 0, \\ e^{Mz_D}[C_1 z_D + C_2], & if \;\; \Delta = 0, \end{cases} \quad \text{(S10)}$$

where $M = \kappa_D/2$; $N = \sqrt{\Delta}$; $N_1 = \sqrt{-\Delta}$; $\Delta = \kappa_D^2/4 + \beta p - \Omega_n^2$. C_1 and C_2 are determined on the basis of Eqs. (S6a) and (S9b), then substituting them into Eq. (S10) yields

$$\mathcal{H}_n = \begin{cases} \cos(\omega_n \alpha_z) \frac{(M+N)\exp[2N(\alpha_z+b_D)+(M-N)z_D]-(M-N)\exp[(M+N)z_D]}{(M+N)\exp[2N(\alpha_z+b_D)+(M-N)\alpha_z]-(M-N)\exp[(M+N)\alpha_z]}, & if \;\; \Delta > 0, \\ \cos(\omega_n \alpha_z)\exp(Mz_D - M\alpha_z) \frac{[N_1 \tan(N_1(\alpha_z+b_D))-M]\sin(N_1 z_D)+[M\tan(N_1(\alpha_z+b_D))+N_1]\cos(N_1 z_D)}{[N_1 \tan(N_1(\alpha_z+b_D))-M]\sin(N_1 \alpha_z)+[M\tan(N_1(\alpha_z+b_D))+N_1]\cos(N_1 \alpha_z)}, & if \;\; \Delta < 0, \\ \cos(\omega_n \alpha_z)\exp(Mz_D - M\alpha_z) \frac{1+M(\alpha_z+b_D)-Mz_D}{1+M(\alpha_z+b_D)-M\alpha_z}, & if \;\; \Delta = 0. \end{cases} \quad \text{(S11)}$$

S3: Evaluation for eigenvalues ω_n

Substituting Eqs. (5) and (7) into Eq. (S6b) yields

$$\begin{cases} -\omega_n \tan(\omega_n \alpha_z) = \frac{1-\exp(-2Nb_D)}{1/(\mathrm{M}-\mathrm{N})-\exp(-2Nb_D)/(\mathrm{M}+\mathrm{N})}, & if \;\; \Delta > 0 \\ -\omega_n \tan(\omega_n \alpha_z) = M + \frac{[N_1 \tan(N_1(\alpha_z+b_D))-M]N_1-[M\tan(N_1(\alpha_z+b_D))+N_1]N_1\tan(N_1\alpha_z)}{[N_1\tan(N_1(\alpha_z+b_D))-M]\tan(N_1\alpha_z)+[M\tan(N_1(\alpha_z+b_D))+N_1]}, & if \;\; \Delta < 0 \\ \omega_n = (\kappa_D^2/4 + (\beta-1)p)^{1/2}. & if \;\; \Delta = 0 \end{cases} \quad \text{(S12)}$$

The eigenvalues of the finite cosine Fourier transform ω_n is the positive root of Eq. (S12). For $\Delta > 0$ the solution domain of ω_n is separated into an infinite series of sub-domain intervals with a period of π, i.e., $\left(\frac{(2i-1)\pi}{2\alpha_z}, \frac{(2i+1)\pi}{2\alpha_z}\right)$, $i = 0,1,2,...$. Each solution of ω_n in an individual sub-domain is obtained using the Newton-Raphson method. For $\Delta < 0$ the solution domain of ω_n is separated into an infinite series of irregular sub-domains due to the complex formula on the right side of the equation. These irregular sub-domains can be identified by seeking the singular points of function $\mathrm{F}(\omega) = M + \frac{[N_1\tan(N_1(\alpha_z+b_D))-M]N_1-[M\tan(N_1(\alpha_z+b_D))+N_1]N_1\tan(N_1\alpha_z)}{[N_1\tan(N_1(\alpha_z+b_D))-M]\tan(N_1\alpha_z)+[M\tan(N_1(\alpha_z+b_D))+N_1]} + \omega\tan(\omega\alpha_z)$, where ω is a variable of function $\mathrm{Y} = \mathrm{F}(\omega)$ and the roots of $\mathrm{F}(\omega) = 0$ are eigenvalues ω_n. Specifically, the eigenvalues ω_n are intersection points of curve $\mathrm{Y} = \mathrm{F}(\omega)$ with $\mathrm{Y} = 0$. These intersection points are separated into an infinite series of irregular sub-domains in which every sub-domain is composed of two adjacent singular points ($\omega = \omega_s$) of function $Y = F(\omega)$. The ω_n in an individual sub-domain is obtained using the Newton-Raphson method. It is shown in Eqs. (5) and (7) that the saturated zone solution involving the components of unsaturated zone is only represented in the ω_n terms. Changes of the unsaturated zone parameters lead to different ω_n values, which affect groundwater flow in the saturated zone.

References

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