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Supplement of

Matching the Budyko functions with the complementary evaporation relationship: consequences for the drying power of the air and the Priestley–Taylor coefficient

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S.1 Properties of the function $E/P = B_1('(\Phi_b)$

Putting $y = E/P$ with $0 \leq y \leq 1$, $x = \Phi_b$ and $a = a_0/(1+b)\alpha_w$, function $B_1'$ given by Eq. (20) can be rewritten as:

$$x = a \left[ (y^{-\lambda} - 1)^{-1/\lambda} + by \right].$$

(S1.1)

For $x = 0$, $y$ is obviously equal to 0. When $x$ tends to infinite the result is less evident. Eq. (S1.1) can be rewritten as:

$$\frac{x}{a} - by = \frac{1}{\left(\frac{1}{y^{\lambda-1}}\right)^{1/\lambda}}.$$  

(S1.2)

When $x$ tends to infinite, given that $y$ is limited by 1, the right-hand term of the equation should tend to infinite. This means that $y$ should tend to 1 so that the denominator tends to zero.

The derivative of the function (Eq. S1.1) is given by:

$$\frac{dx}{dy} = a \left[ b + y^{-(\lambda+1)}(y^{-\lambda} - 1)^{-\lambda-1/\lambda} \right].$$

(S1.3)

which can be rewritten as:

$$\frac{dy}{dx} = \frac{1}{a} \left[ b + (1 - y^{\lambda})^{-(1+\lambda)/\lambda} \right]^{-1}.$$  

(S1.4)

Close to $x = 0$, $y$ is close to zero and the derivative can be approximated by:

$$\frac{dy}{dx} \approx \frac{1}{a(1+b)} \left[ 1 - \left( \frac{1+\lambda}{\lambda(1+b)} \right) y^{\lambda} \right] \approx \frac{\alpha_w}{\alpha_0}. \quad (S1.5)$$

If Eq. (22) is taken into account:

$$\frac{\alpha_w}{\alpha_0} = \left[ \frac{1+b(1+x^{\lambda})^{-1/\lambda}}{(1+b)} \right],$$

(S1.6)

which means that $\alpha_w/\alpha_0$ and $dy/dx$ tend to 1 when $x$ tends to zero.
S.2 Properties of the function \( E/E_0 = B'_2(\Phi_0^{-1}) \)

With \( X = \Phi_0^{-1}, Y = E/E_0 \) and the parameter \( a \) defined as above in S1, function \( B'_2 \) (Eq. 21) can be written as

\[
X^{-\lambda} = Y^{-\lambda} - \left( \frac{1}{a} - bY \right)^{-\lambda}.
\]  
(S2.1)

When \( X \) tends to zero, \( Y \) (limited by \( l \)) necessarily tends to zero, and when \( X \) tends to infinite \( Y \) tends to \( 1/(1+b)a = \alpha_w/\alpha_0 \), which is equal to \( l \) according to Eq. (S1.6) (\( x = 1/X = 0 \)).

The derivative of \( B'_2 \) can be written as:

\[
\frac{dY}{dX} = \frac{X^{-\lambda-1}}{Y^{-\lambda-1} + b \left( \frac{1}{a} - bY \right)^{-\lambda-1}} = \frac{1}{(X/Y)^{\lambda+1} + \left[ \frac{1}{(X/Y)^{\lambda+1}} \right]}.
\]  
(S2.2)

When \( X \) tends to zero, \( Y \) also tends to zero and the term into square brackets tends to \( l \) which means that:

\[
\frac{dY}{dX} \rightarrow \left( \frac{Y}{X} \right)^{\lambda+1}.
\]  
(S2.3)

Taking into account Eq. (S2.1), we have:

\[
\left( \frac{Y}{X} \right)^{\lambda+1} = \left[ 1 - \left( \frac{1}{aY} - b \right)^{-\lambda} \right]^{\lambda+1}.
\]  
(S2.4)

which tends to \( l \).

S.3 Transcendental forms of the basic equations \( E/P = B_1(\Phi_p) \) and \( E/E_p = B_2(\Phi_p^{-1}) \)

Eqs. (4) and (5) have the same following form:

\[
y = \left( 1 + x^{-\lambda} \right)^{-1/\lambda},
\]  
(S3.1)

with \( x = \Phi_p \) and \( y = E/P \). Eq. (S3.1) can be also written as:

\[
x = \left( y^{-\lambda} - 1 \right)^{-1/\lambda}.
\]  
(S3.2)

With similar notations, Eq. (23) can be written as:

\[
y + \left( y^{-\lambda} - 1 \right)^{-1/\lambda} = x + \left( 1 + x^{-\lambda} \right)^{-1/\lambda}.
\]  
(S3.3)

Eq. (S3.3) is equivalent to \( y + x = x + y \), which means that S3.1 or S3.2 are solutions of Eq. (S3.3).

A similar reasoning can be conducted with Eq. (24), which can be written with \( X = \Phi_p^{-1} \) and \( Y = E/E_p \):

\[
\left[ 1 - Y + \left( 1 + X^{-\lambda} \right)^{-1/\lambda} \right]^{-\lambda} = Y^{-\lambda} - X^{-\lambda}.
\]  
(S3.4)

Given that Eq. (S3.1) is verified by \( X \) and \( Y \):

\[
Y^{-\lambda} = 1 + X^{-\lambda}.
\]  
(S3.5)

Eq. (S3.4) is equivalent to \( l = l \), which means that Eq. (S3.1) or (S3.2) is solution of Eq. (S3.4).
S.4 Calculations made with the Fu-Zhang equation

The Fu-Zhang equation is written as:

\[
\frac{E}{P} = 1 + \phi_p - \left[ 1 + \left( \phi_p \right)^\omega \right]^{\frac{1}{\omega}}.
\]  

(S4.1)

First, we study the feasible domain of the drying power of the air \( E_a \) and the correspondence with the evaporation rate \( E \).

5 Inserting Eq. (S4.1) into Eq. (9) yields:

\[
\frac{E_a}{E_p} = D\left( \phi_p^{-1} \right) = \left( 1 + \frac{\Delta}{\gamma} \right) \left( 1 - \frac{1}{(1+b)\alpha_w} \left( 1 + b \left( 1 + \phi_p^{-1} - (1 + \phi_p^{-\omega})^{\frac{1}{\omega}} \right) \right) \right).
\]  

(S4.2)

The limits given in Eqs. (11), (12) and (13) are independent from the Budyko function used. Consequently \( D^* \) remains unchanged:

\[
D^* = \frac{b}{(1+b)\alpha_w} \left( 1 + \frac{\Delta}{\gamma} \right).
\]  

(S4.3)

10 Using a similar reasoning as in Eqs (14), (15), (16) and (17), we obtain:

\[
d^* = 2\frac{1}{\omega} - 1,
\]  

(S4.4)

\[
\omega = \frac{\ln 2}{\ln (d^*-1)},
\]  

(S4.5)

\[
\delta^* = \left( 1 + \frac{\Delta}{\gamma} \right) \frac{b}{(1+b)\alpha_w} \left( 1 - 2^{-\frac{1}{\omega}} \right) = D^* d^*.
\]  

(S4.6)

Second, we link the Priestley-Taylor coefficient \( \alpha_0 \) to the Fu-Zhang shape parameter \( \omega \). Substituting \( E_p \) in Eq. (S4.1) by its value given by Eq. (18) and putting \( \phi_i = E_i/P \) gives:

\[
\frac{E}{P} = 1 + \frac{(1+b)\alpha_w}{\alpha_0} \phi_0 - b \frac{E}{P} - \left[ 1 + \left( \frac{(1+b)\alpha_w}{\alpha_0} \phi_0 - b \frac{E}{P} \right) \right]^{\frac{1}{\omega}}.
\]  

(S4.7)

Eq. (S4.7) can be rewritten as:

\[
\left[ 1 + (1+b) \left( \frac{\alpha_w}{\alpha_0} \phi_0 - \frac{E}{P} \right) \right]^{\omega} = 1 + \left[ (1+b) \frac{\alpha_w}{\alpha_0} \phi_0 - b \frac{E}{P} \right]^{\omega}.
\]  

(S4.8)

An equation similar to Eq. (21) can be obtained expressing \( E/E_0 \) as a function of \( \phi_i^{-1} = P/E_i^* \):

\[
\left[ 1 + \frac{(1+b)\alpha_w}{\phi_i^{-1} - \frac{E}{E_0}} \right]^{\omega} = 1 + \left( \frac{1}{\phi_i^{-1}} \right)^{\omega} \left[ \frac{(1+b)\alpha_w}{\alpha_0} - b \frac{E}{E_0} \right]^{\omega}.
\]  

(S4.9)

Eqs. (S4.8) and (S4.9) obtained from the Fu-Zhang formulation correspond respectively to \( E/P = B_1 \phi_i \) (Eq. 20) and \( E/E_0 = B_2 \phi_i^{-1} \) (Eq. 21) obtained with the Turc-Mezentsev equation.

Using a similar reasoning as in Eq. (22), the expression of \( \alpha_0 \) can be inferred by matching Eqs. (S4.8) and (S4.1):

for a given value of the aridity index \( \phi \), we have the same value of \( E/P \). This leads to:

\[
\left( 1 + b \right) \left[ \left( \frac{\alpha_w}{\alpha_0} - 1 \right) \phi + \left( 1 + \phi^\omega \right)^{\frac{1}{\omega}} - b \right]^{\omega} = 1 + \left[ \left( \frac{(1+b)\alpha_w}{\alpha_0} - b \right) \phi + b \left( 1 + \phi^\omega \right)^{\frac{1}{\omega}} - b \right]^{\omega}.
\]  

(S4.10)

Eq. (S4.10) is equivalent to Eq. (22), but with a transcendental form. It can be resolved numerically and Fig. (S1) shows the variation of the Priestley-Taylor coefficient \( \alpha_0 \) as a function of the aridity index \( \phi \) for different values of the \( \omega \) parameter. The shape of the curves is very similar to those of Fig. (5a) obtained with the parameter \( \lambda \) of the Turc-Mezentsev function.
Figure S1: Variation of the Priestley-Taylor coefficient $\alpha_0$ with $b = 1$ as a function of the aridity index $\Phi$ for different values of the shape parameter $\omega$ of the Fu-Zhang function. The bold lines indicate the limits of the feasible domain.
S.5 Results obtained with the Turc-Mezentsev function making $b = 4.5$ instead of $b = 1$

Figure S2: Variation of the Priestley-Taylor coefficient $\alpha_0$ (Eq. (22) with $b = 4.5$ and $\alpha_w = 1.26$): (a) as a function of the aridity index $\Phi$ for different values of the shape parameter $\lambda$ of the Turc-Mezentsev function; (b) as a function of $\lambda$ for different values of the aridity index $\Phi$. The bold lines indicate the limits of the feasible domain.